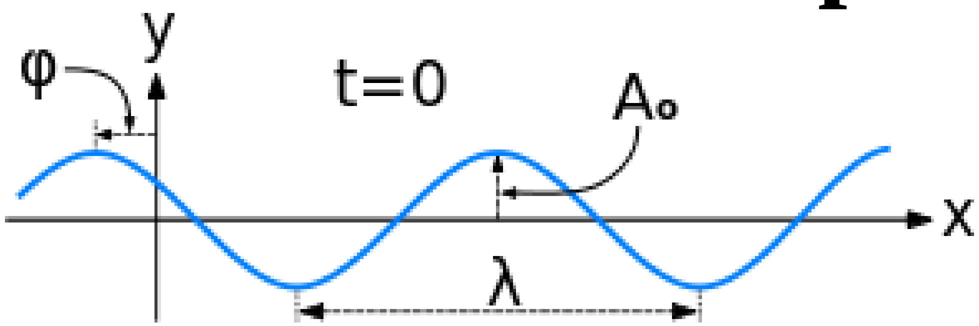
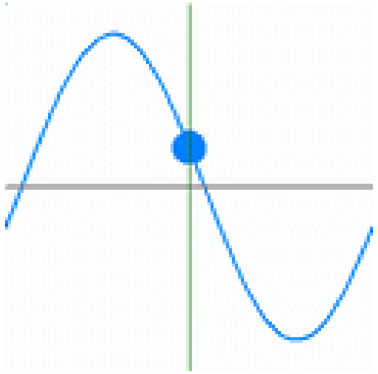
Transverse Wave Motion

14TH SEPTEMBER 2020

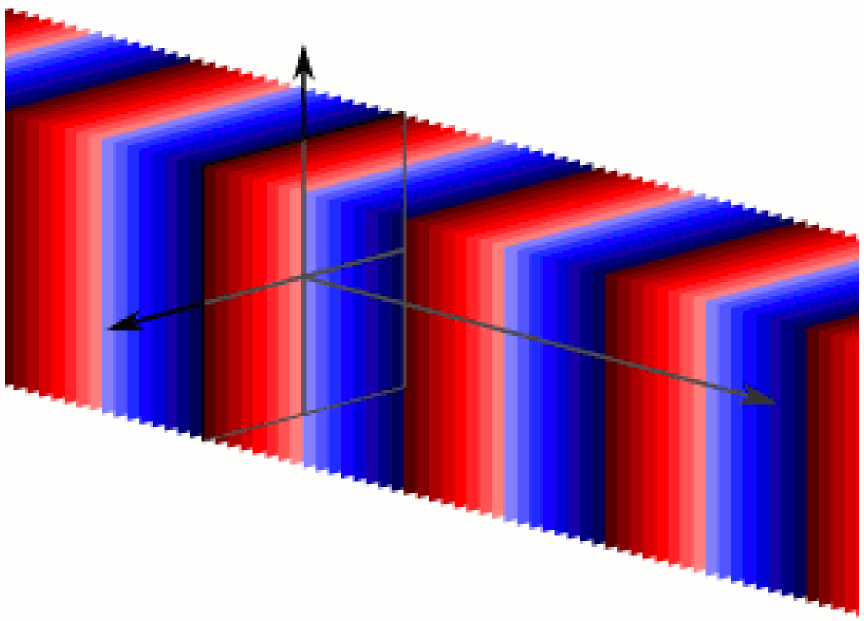
Concept of Plane wave



A typical representation of a wave propagating in space



The propagating wave causing the individual oscillators in the medium to oscillate



Overall oscillation of the oscillators with the same phase in the same plane forms the travelling plane wave.
Note : each color represents a different phase of the wave.

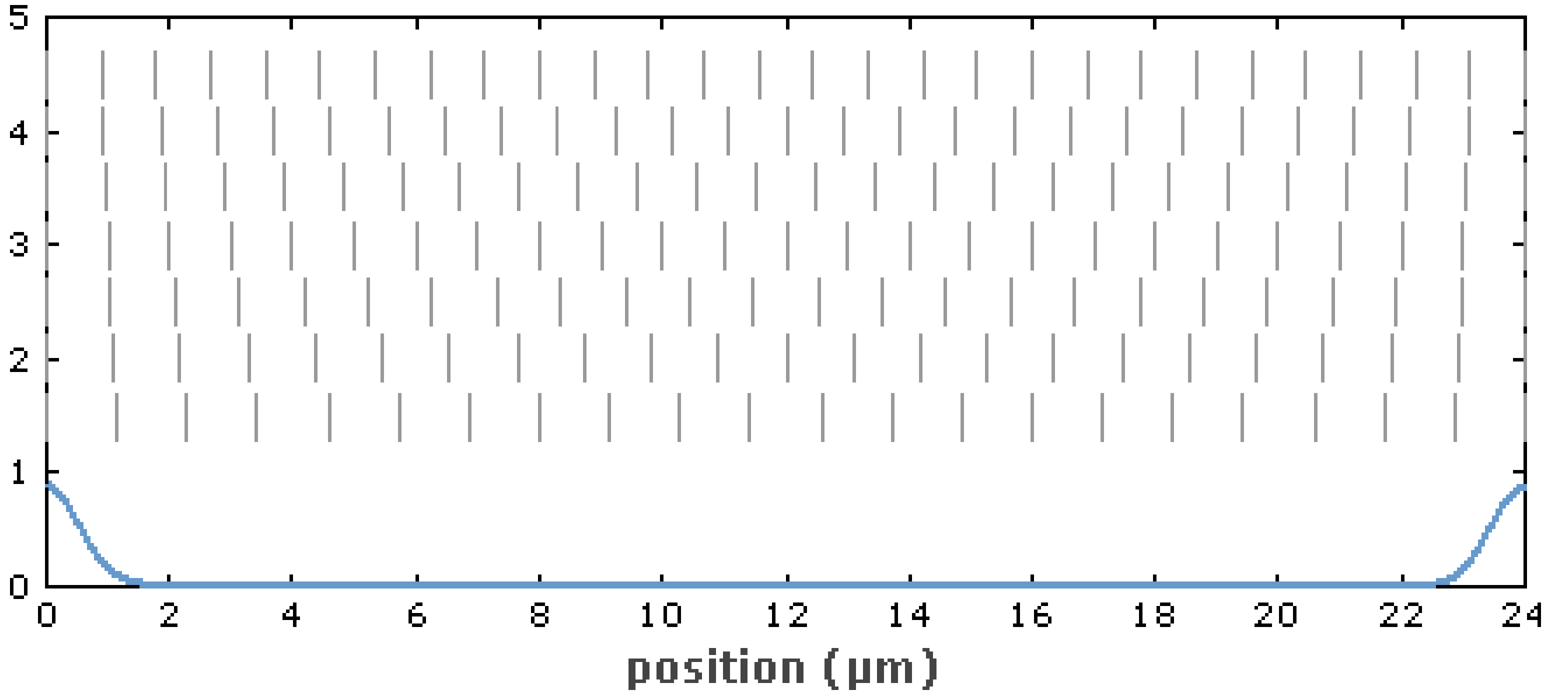
Velocities in wave motion

- For a mechanical wave, the individual oscillators which make up the medium **do not progress** through the medium with the waves.
- The **three velocities in wave motion** are

  1. The particle velocity
  2. The wave or phase velocity
  3. The group velocity

- For a **monochromatic wave**, the group velocity and the wave velocity are identical.

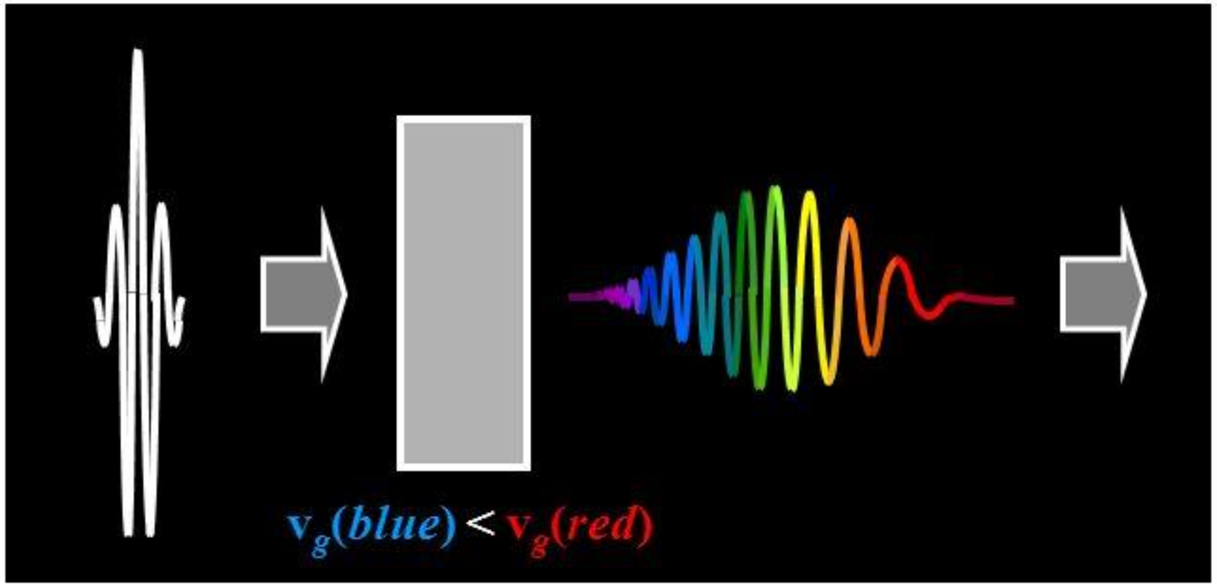
Example of the group velocity



- Propagation of a light pulse in a **dispersive medium**. Note that the phase fronts of different frequency components propagate with different velocities, and the pulse propagates with the **group velocity**, which is lower than all the phase velocities;
- No temporal broadening due to the propagation taking place in a **nondispersive medium**.

Group velocity **dispersion** is the variation of group velocity with wavelength

GVD means that the group velocity will be different for different wavelengths in the pulse. So GVD will lengthen a pulse in time.



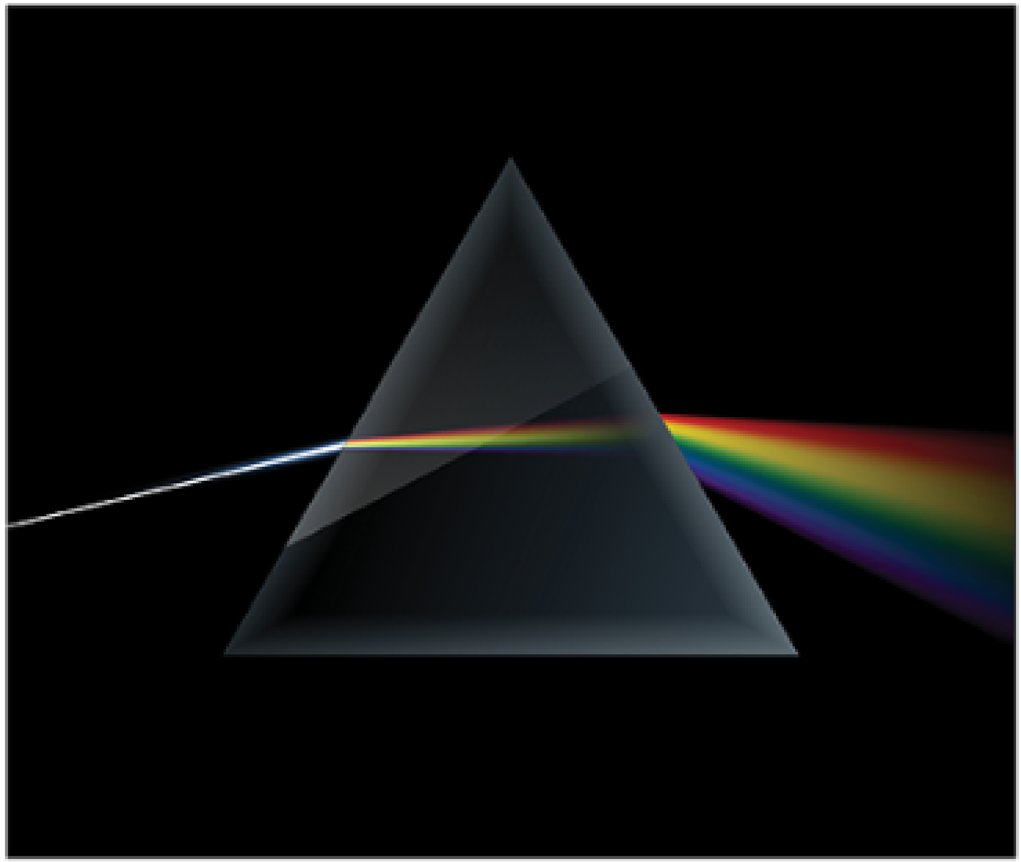
Because short pulses have such large ranges of wavelengths, GVD is a bigger issue than for nearly monochromatic light.

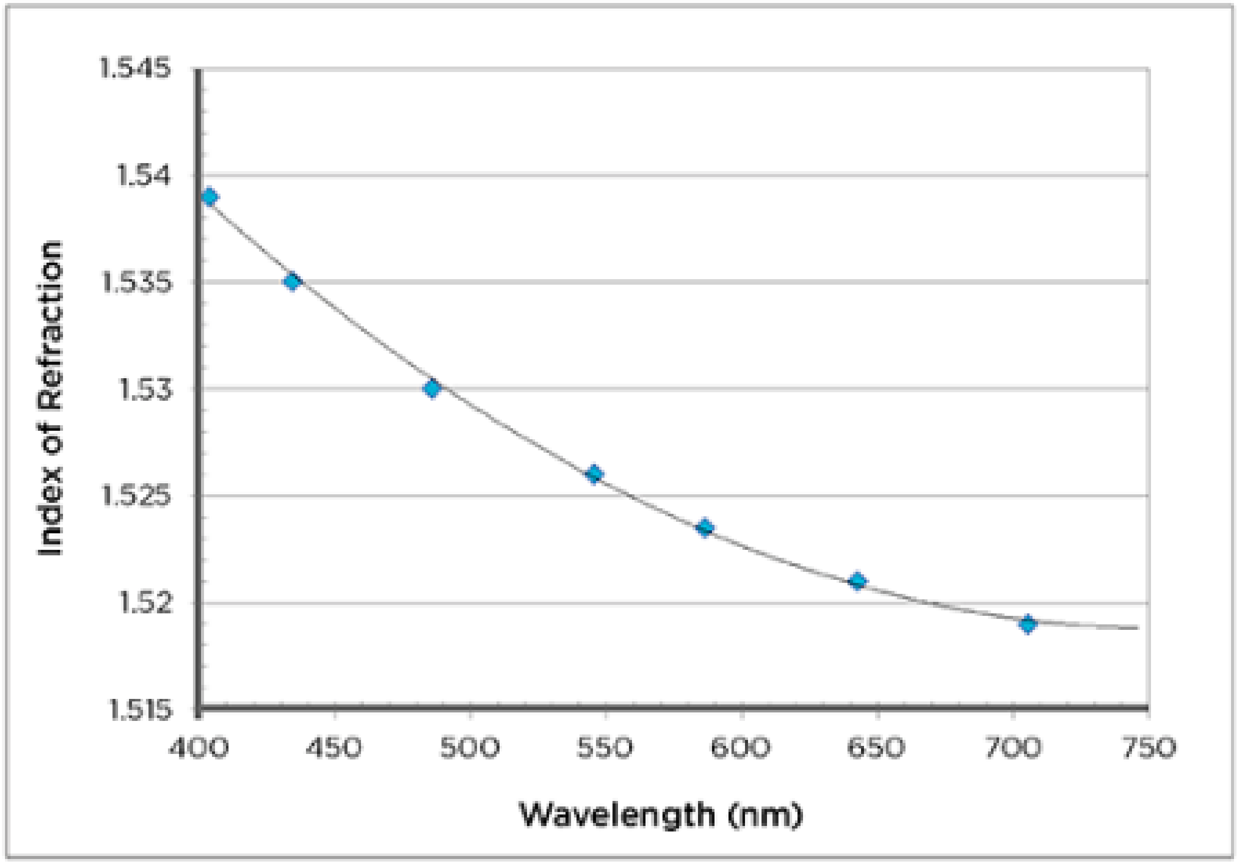
28

- White light pulse consists of an infinite fine spectrum of frequencies and moving with a group velocity.
- The wave velocity of each wavelength in the group is different.
- The phenomenon is known as **dispersion**.

(GVD = Group Velocity Dispersion)

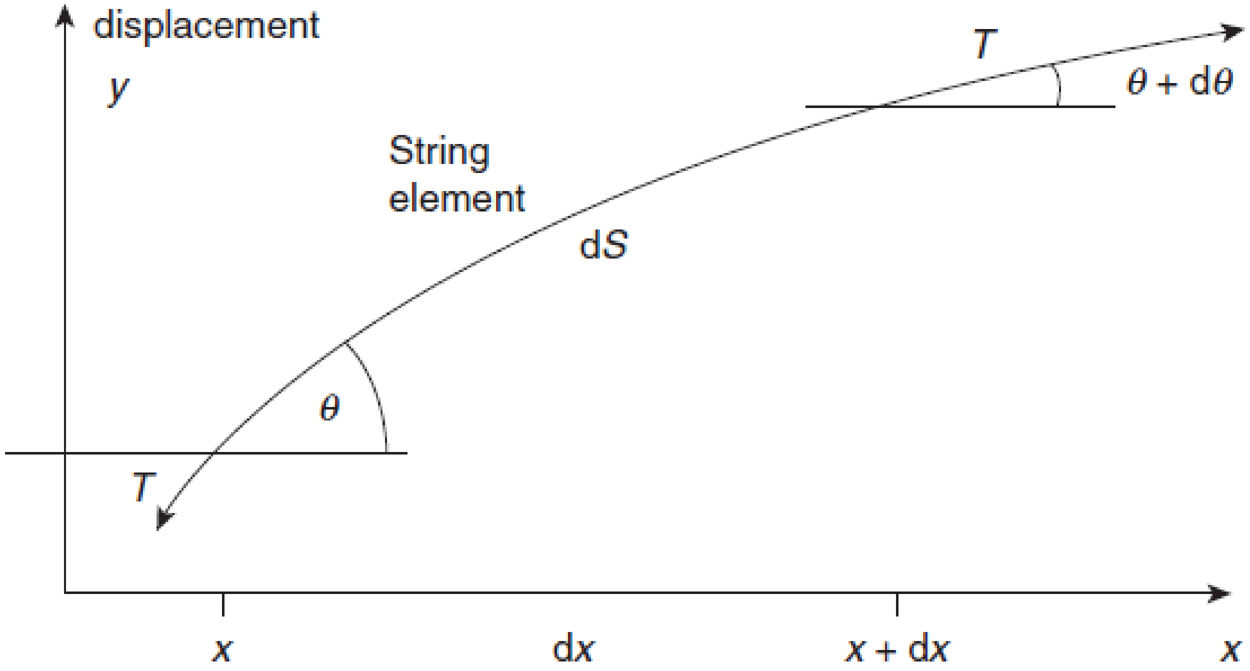
Example of dispersion





The index of refraction for a glass material changing over the visible wavelength spectrum. The use of an optical prism shows the effect of this change index across the visible spectrum as white light is split into individual wavelengths and colors.

The transverse wave equation on a string



- A very short section of a uniform string has a vertical displacement.
- The wave equation relates the vertical displacement y to time t and position x.
- Given that the linear density ρ and a constant tension T along a slightly extensible string, the wave equation is given as

$$\frac{\partial^2 y}{\partial^2 x} = \frac{\rho}{T}\frac{\partial^2 y}{\partial^2 t} = \frac{1}{c^2}\frac{\partial^2 y}{\partial^2 t}$$

Ideal string constraints

In order for the wave equation to apply to the waves in a string, it must meet certain constraints. For an ideal string, it is assumed that

1. The string is perfectly uniform with a constant mass per unit length and is perfectly elastic with no resistance to bending.

2. The string tension is presumed to be large enough so that **gravity can be neglected**.

3. Small segments of the string are presumed to move transversely in a plane perpendicular to the string, and that the displacements and slopes of segments of the string are small.

http://hyperphysics.phy-astr.gsu.edu/hbase/Waves/waveq.html#c2

How to obtain the wave equation

- The length of the string short section is given by dS which can be written as

$$dS = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \approx dx$$; the slope is presumably small.

- By applying the Newton's second law of motion, the horizontal force components on the element are cancelled ;i.e., $T\cos(\theta + d\theta) \approx T\cos\theta$.

- And, the vertical force components results in

$$T\left[\left(\frac{\partial y}{\partial x}\right)_{x+dx} - \left(\frac{\partial y}{\partial x}\right)_{x}\right] = \rho dx \frac{\partial^2 y}{\partial^2 t}$$

The wave equation of a wave on a string

- According to the definition of the partial derivative, we have

$$T\left[\left(\frac{\partial y}{\partial x}\right)_{x+dx}-\left(\frac{\partial y}{\partial x}\right)_{x}\right]=Tdx\frac{\partial^2 y}{\partial^2 x}=\rho dx\frac{\partial^2 y}{\partial^2 t}$$

- Finally, the wave equation of a string is found to be

$$\frac{\partial^2 y}{\partial^2 x}=\frac{\rho}{T}\frac{\partial^2 y}{\partial^2 t}=\frac{1}{c^2}\frac{\partial^2 y}{\partial^2 t}$$

; c is the wave velocity of the wave on the string.

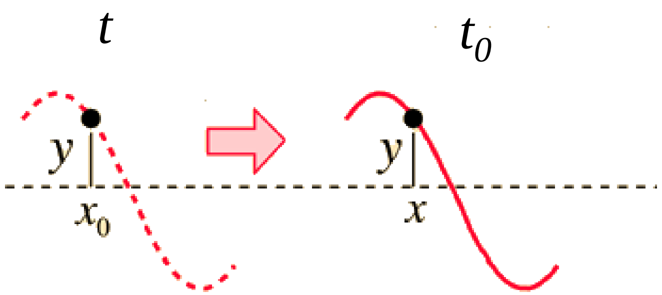
Solution of the wave equation : travelling wave

- The solution of the wave equation is a function of the variables x and t.
- The right and left travelling (or progressive) waves can be written as

$$y(x,t) = a\sin(\omega t - kx);\ \text{moving to the right: +x direction}$$

$$y(x,t) = a\sin(\omega t + kx);\ \text{moving to the left: -x direction}$$

- Note :



- Initially, a wave function is given as $y = a\sin(\omega t_0 - kx_0)$.
- At a later time t ($>t_0$), the wave function becomes $y = a\sin(\omega t - kx)$.
- $\therefore\ \omega t_0 - kx_0 = \omega t - kx$ and this leads to $x > x_0$. This suggests the **right** moving waveform.

http://hyperphysics.phy-astr.gsu.edu/hbase/Waves/wavsol.html#c3

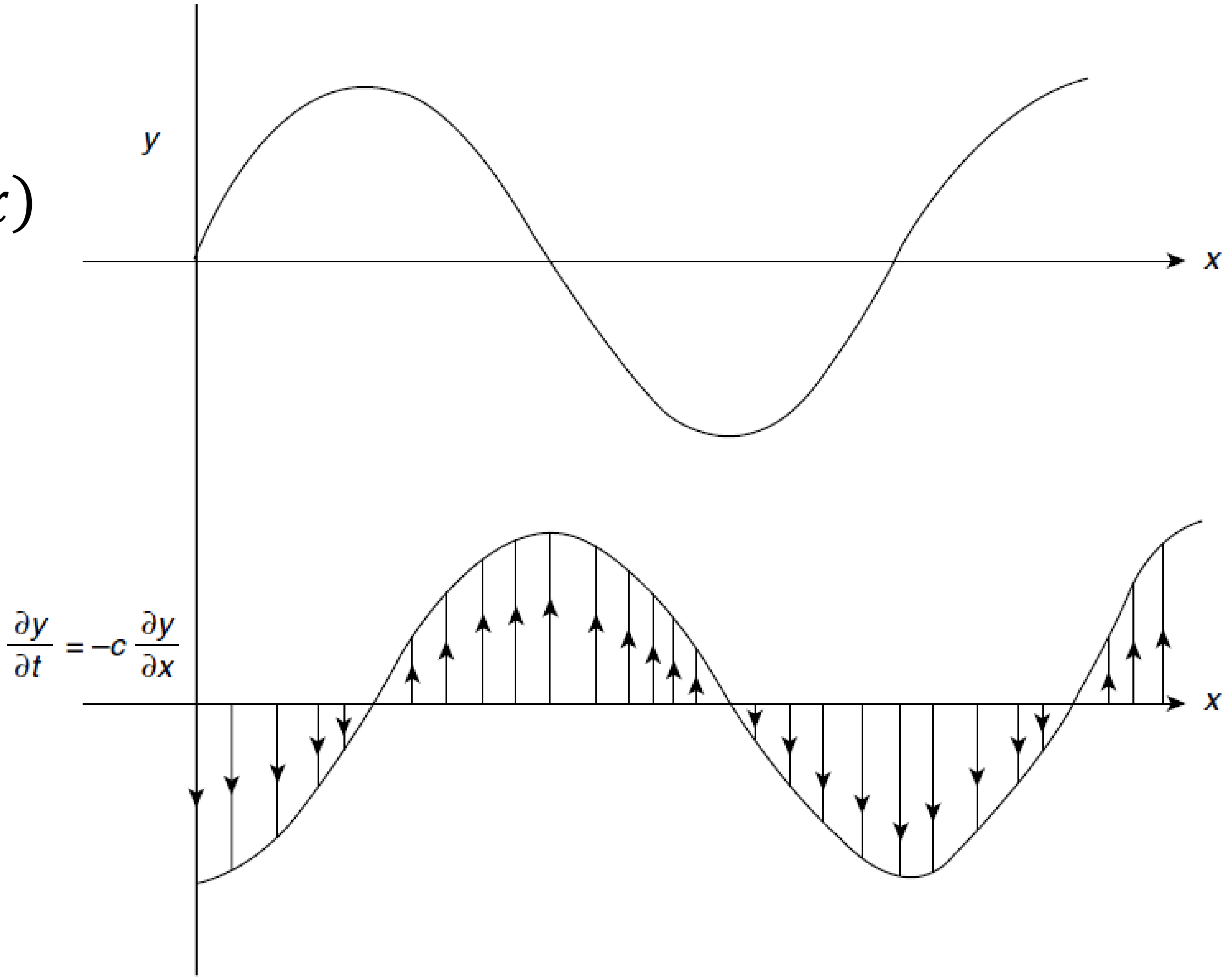
The wave velocity and the particle velocity

- Let's consider the wave function : $y = a\sin(\omega t - kx)$.
- We found that $\frac{d^2}{dt^2}y = \boxed{}$ and $\frac{d^2}{dx^2}y = \boxed{}$
- By substituting these two equations into the wave equation, $c = \boxed{}$ **clearly represents the wave velocity**.
- We also found that $\frac{\partial y}{\partial t} = \boxed{}$ and $\frac{\partial y}{\partial x} = \boxed{}$.
- **This leads to the particle velocity** $\frac{\partial y}{\partial t} = -c\frac{\partial y}{\partial x}$ which is the product of the wave velocity and the gradient of the wave profile preceded by a negative sign for a right-going wave.

Displacement

$y = a\sin(\omega t - kx)$

Particle velocity



The magnitude and direction of the particle velocity $\frac{\partial y}{\partial t} = -c\frac{\partial y}{\partial x}$ at any point x is shown by an arrow in the right-going sine wave above

Characteristic impedance

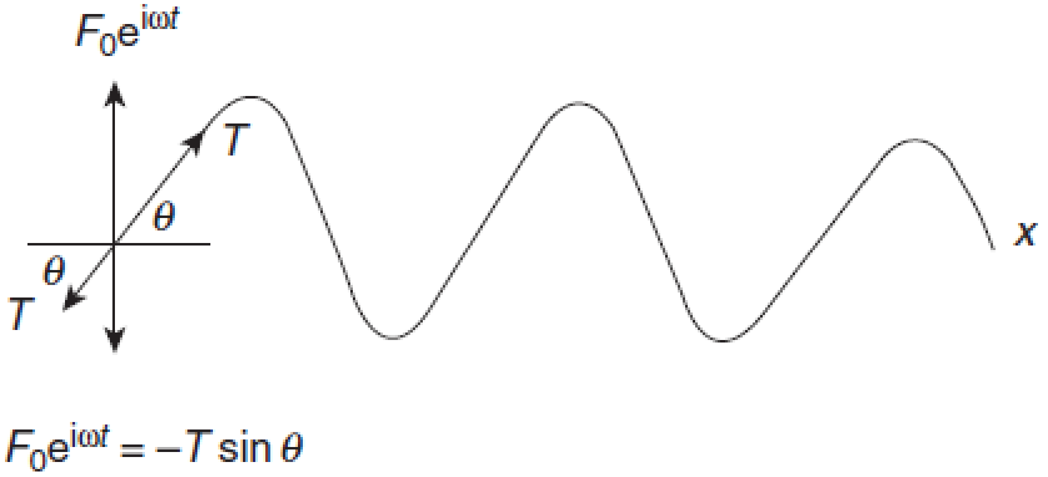
- Any medium through which waves propagate will present an impedance to those waves.
- The impedance to progressive transverse waves is defined as the **transverse impedance**,

$$Z = \frac{\text{transverse force}}{\text{transverse velocity}} = \frac{F}{v}$$

- The velocity v here is the particle velocity **NOT** the wave velocity.
- For a harmonic force, the ratio of force to velocity describes how hard to move the object – this depends both on how much inertia (e.g. linear density) the object has and on how strong the restoring force (e.g. tension) is.
- The impedance, represented a medium characteristic, can be either **real (for lossless case) or complex (for loss case)**.

Characteristic impedance of a string

The string as a forced oscillator with a vertical force $F_0\exp(i\omega t)$ driving at one end.



- For a minimum value of the particle velocity, the constant tension T in the string is balanced by the force at the end of the string.
- This gives

$$F_0 e^{i\omega t} = -T\sin\theta \approx -T\tan\theta = -T\left(\frac{\partial y}{\partial x}\right);\ \text{small}\ \theta$$

Characteristic impedance of a string (cont.)

- If the the displacement of the progressive waves is given as $y = Ae^{i(\omega t - kx)}$
- The minimum velocity of the string element is obtained from $F_0 e^{i\omega t} = -T\left(\frac{\partial y}{\partial x}\right)$
- Therefore, the displacement at x = 0 is

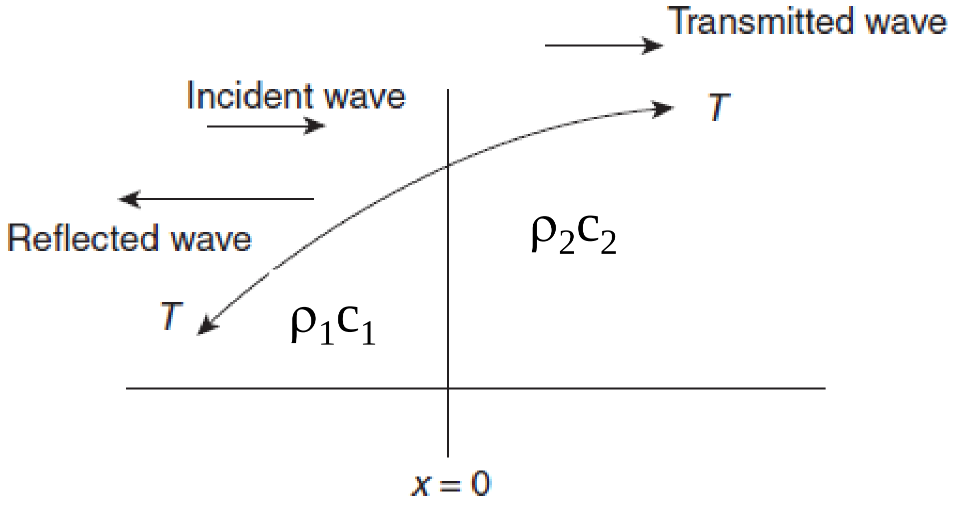
$$y = \boxed{}$$

- The velocity is found to be $v = \dot{y} = \boxed{}$

- This leads to an alternative representation of the **transverse string characteristic impedance as the ratio of the tension to the wave velocity**

$$Z = \frac{F}{v} = \frac{T}{c} = \rho c$$

Reflection and transmission of wave on a string at a boundary



Waves on a string impedance $\rho_1 c_1$ reflected and transmitted at the boundary x = 0 where the string changes to impedance $\rho_2 c_2$

Boundary conditions at x = 0 :

1. **geometrical condition** : displacement is the same immediately to the left and right at x = 0 for all time (a continuity of the string)

2. **dynamical condition** : a continuity of the transverse force $T(\partial y \partial / x)$ at x = 0, and therefore a continuous slope.

Incident, reflected and transmitted waves

$$\text{Incident wave}: y_i(x,t) = A_1 \exp(\omega t - k_1 x)$$

$$\text{Reflected wave}: y_r(x,t) = \boxed{}$$

$$\text{Transmittedwave}: y_t(x,t) = \boxed{}$$

This suggests that at the boundary the incident wave is partly transmitted and partly reflected due to different impedances of the media.

Determination of the reflection and transmission amplitude coefficients

- According to the boundary conditions (at x =0 for all t)

$$\text{BC 1}: \; y_i + y_r = y_t$$

This condition suggests that there is a continuity of the displacement.

$$\text{BC 2}: \; T\frac{\partial}{\partial x}\left(y_i + y_r\right) = T\frac{\partial}{\partial x}y_t$$

This condition suggests that the transverse forces have to be equal so as to prevent an infinite acceleration of an infinitesimal segment of string at the interface.

Reflection and transmission coefficients

Reflection coefficient of amplitude, $\frac{B_1}{A_1} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$

transmission coefficient of amplitude, $\frac{A_2}{A_1} = \frac{2Z_1}{Z_1 + Z_2}$

- Note : these coefficients are independent of ω and hold for waves of all frequencies.

Reflections of string at fixed end (Z = ∞) and at free end (Z = 0)

At a fixed (hard) boundary, the displacement remains zero and the reflected wave changes its polarity (undergoes a 180^o *phase change)*

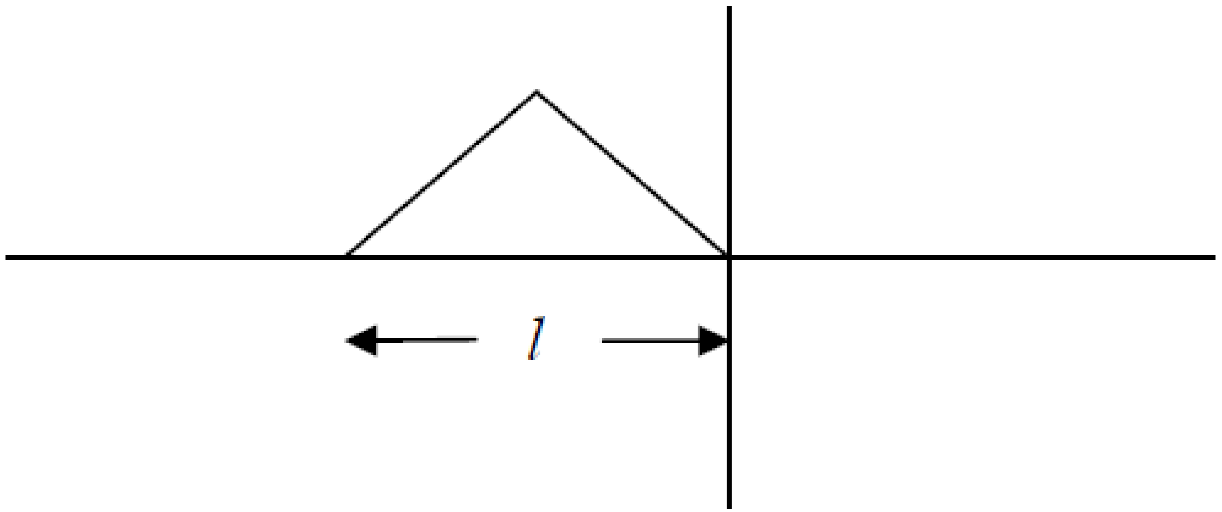
At a free (soft) boundary, the restoring force is zero and the reflected wave has the same polarity (no phase change) as the incident wave

https://www.acs.psu.edu/drussell/demos/reflect/reflect.html

Example 1

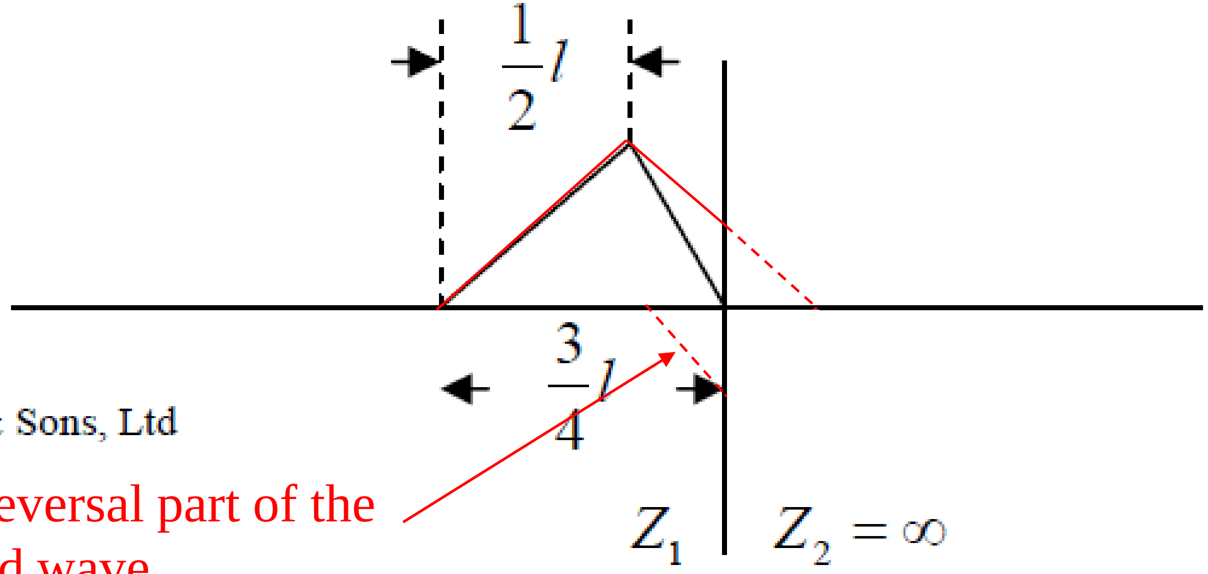
A triangular shaped pulse of length l is reflected at the fixed end of the string on which it travels ($Z_2 = \infty$). Sketch the shape of the pulse after a length (a) $l/4$ (b) $l/2$ (c) $3l/4$ and (d) l of the pulse has been reflected.

The pulse shape before reflection is given by the graph below:



How to obtain a reflected pulse

(a) $\Delta l = l/4$

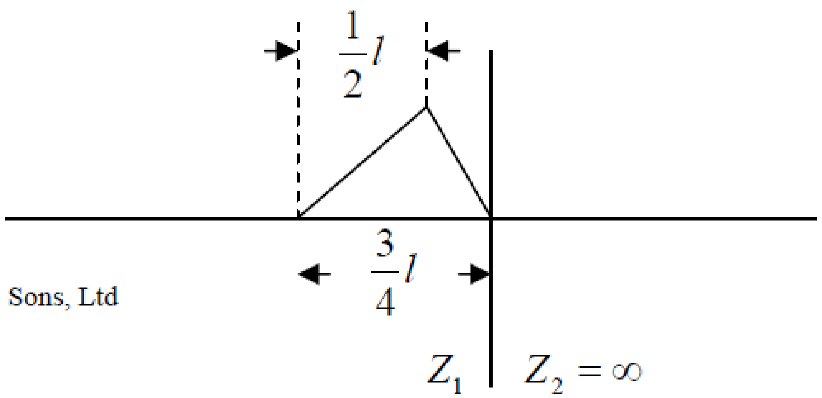


© 2008 John Wiley & Sons, Ltd

Phase reversal part of the reflected wave

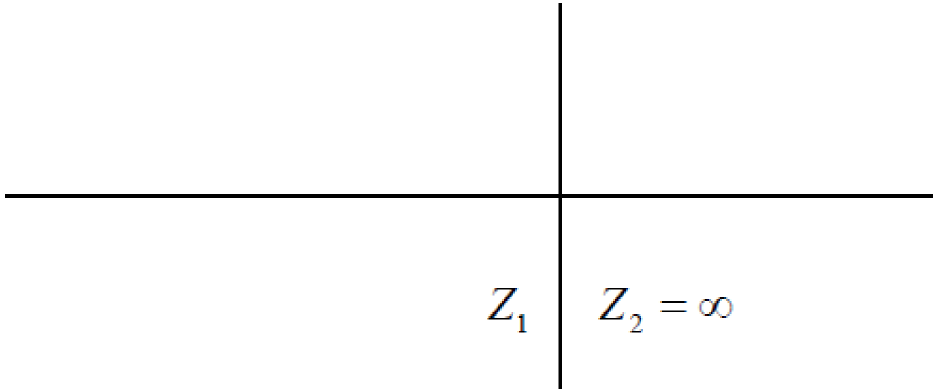
Solution

(a) $\Delta l = l/4$

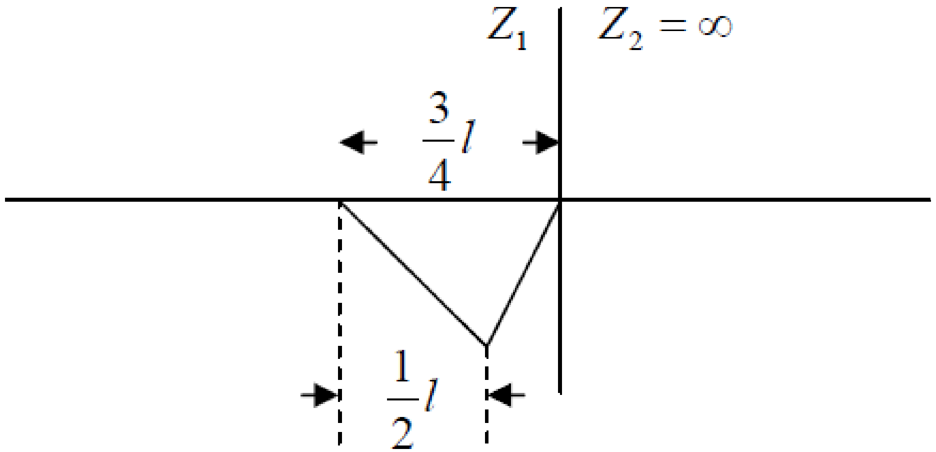


© 2008 John Wiley & Sons, Ltd

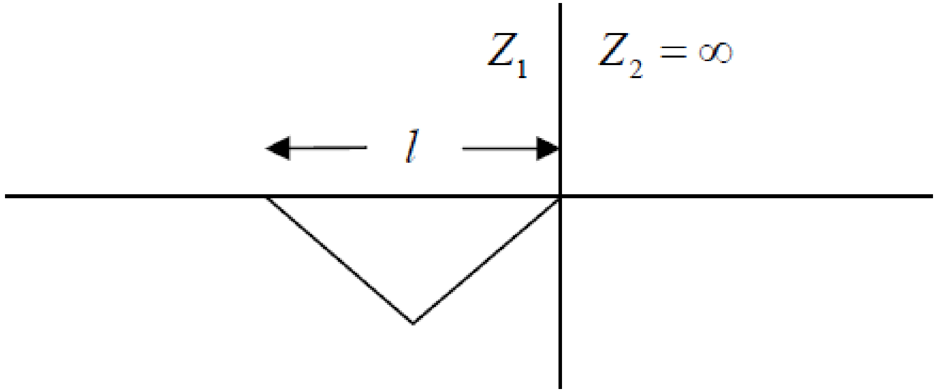
(b) $\Delta l = l/2$



(c) $\Delta l = 3l/4$



(d) $\Delta l = l$



An incident wave travelling from a high density (low wave speed) region towards a low density (high wave speed) region.	An incident wave travelling from a low density (high wave speed) region towards a high density (low wave speed) region.
• *How do the polarities of the reflected and transmitted waves compare to the polarity of the incident wave?*	• *How do the polarities of the reflected and transmitted waves compare to the polarity of the incident wave?*

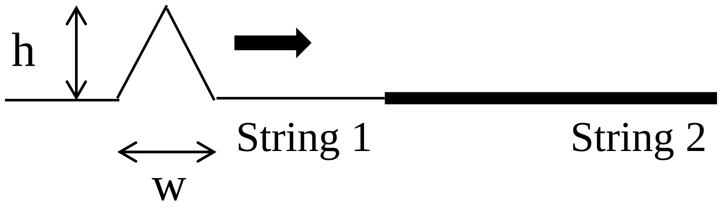
Example 2 : triangle wave

- Consider the waveform shown below heading towards a boundary between two strings.
- Let string 1 have a mass per unit length of ρ_1 = 0.1 kg/m.
- Let string 2 have a mass per unit length of ρ_2 = 0.2 kg/m.
- Let the tension of the string be T = 20 N.

a) Find the wave speed in each string.

b) Find the impedance of each string.

c) Find the shape of the reflected and transmitted waves and sketch them just after the incident wave completely passed the boundary.



Solution

a) the wave speed in each string can be calculated from $\sqrt{T/\mu}$: c_1 = 14.1 m/s and c_2 = 10 m/s.

b) the impedance of each string can be calculated from ρc : $\rho_1 c_1$ = 1.4 kg·m/s, $\rho_2 c_2$ = 2.0 kg·m/s.

c) The reflected and transmitted waves are triangle.

The amplitude of the reflected wave is found from $$\frac{B_1}{A_1} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \rightarrow B_1 = h\left(\frac{\rho_1 c_1 - \rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2}\right) = -0.18h$$

The width of the reflected wave becomes……….................... to the width of the incident wave.

The amplitude of the transmitted wave is found from $$\frac{A_2}{A_1} = \frac{2Z_1}{Z_1 + Z_2} \rightarrow A_2 = h\left(\frac{2\rho_1 c_1}{\rho_1 c_1 + \rho_2 c_2}\right) = 0.82h$$

The width of the transmitted wave is found to be………………than the width of the incident wave.

Reflection and transmission of energy

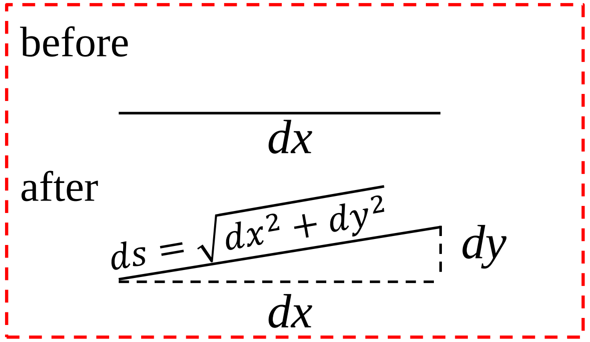
- Now, let's consider what happens to the energy flow in a wave when it meets a boundary between two media of **different impedance values**.
- Generally, **the energy flow or the rate at which energy is being carried** along a unit length, mass ρ, of the string as a simple harmonic oscillator of maximum amplitude A with travelling wave velocity of c is

$$\left(energy \times velocity\right) = \frac{1}{2}\rho\omega^2 A^2 c$$

- This can be shown that the energy arriving at the boundary x = 0 is equal to the energy leaving the boundary. In other words, the energy is conserved.

Derive the energy flow of a wave on a string

- Suppose the vertical displacement of a string element with mass dm is given by dy.
- The kinetic energy (dK) of the string element is written as $\frac{1}{2}dm\left(\frac{dy}{dt}\right)^2 = \frac{1}{2}\rho\left(\frac{dy}{dt}\right)^2 dx.$
- The elastic potential energy (dU) is equal to the work done by the tension T to stretch the string and can be written as $dU = dW = T(ds - dx)$; where $(ds - dx)$ represents the stretched length of the string under the tension.
- Since $ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \approx dx\left(1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 + \cdots\right)$; therefore $(ds - dx) = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$
- Finally, the elastic potential energy $dU = \frac{1}{2}T\left(\frac{dy}{dx}\right)^2 dx.$
- The total energy can be written as $dE = \frac{1}{2}\rho\left(\frac{dy}{dt}\right)^2 dx + \frac{1}{2}T\left(\frac{dy}{dx}\right)^2 dx$



Derive the energy flow of a wave on a string (cont.)

- Recall the total energy : $dE = \frac{1}{2}\rho\left(\frac{dy}{dt}\right)^2 dx + \frac{1}{2}T\left(\frac{dy}{dx}\right)^2 dx$
- Suppose the displacement of the string element follows a harmonic function such as $y = Asin(\omega t - kx)$.
- $\left(\frac{\partial y}{\partial t}\right)^2 = \omega^2 A^2 cos^2(\omega t - kx)$ and $\left(\frac{\partial y}{\partial x}\right)^2 = k^2 A^2 cos^2(\omega t - kx)$
- **Let's try to determine the total energy over a wavelength and then calculate the energy flow over a period or power of the wave on a string!**

Solution

- As the displacement is written in the harmonic function given in the previous slide, the total energy becomes

$$dE = \frac{1}{2}\rho\, \omega^2 A^2 cos^2(\omega t - kx)\, dx + \frac{1}{2} T k^2 A^2 cos^2(\omega t - kx) dx$$

- Due to $T = c^2\rho = \left(\frac{\omega}{k}\right)^2 \rho$, $dE = \rho\, \omega^2 A^2 cos^2(\omega t - kx)\, dx$
- Over a wavelength, the total energy is written as $E_\lambda = \int_0^\lambda \rho\omega^2 A^2 cos^2(\omega t - kx)\, dx = \frac{1}{2}\rho\omega^2 A^2 \lambda$.
- Finally, the energy flow over a period T or power is given $P = \frac{E_\lambda}{T} = \frac{1}{2}\rho\omega^2 A^2 c;\ c = \lambda/T$.

The conservation of energy

- All energy arriving at the boundary in the incident wave leaves the boundary in the reflected and transmitted waves.

energy arriving = energy leaving

- Note : The proof requires the transformation of B_1 and A_2 to be in terms of A_1.

The reflected and transmitted intensity coefficients

$$\frac{\text{Reflected Energy}}{\text{Incident Energy}} = \frac{Z_1 B_1^2}{Z_1 A_1^2} = \left(\frac{B_1}{A_1}\right)^2 = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$$

$$\frac{\text{Transmitted Energy}}{\text{Incident Energy}} = \frac{Z_2 A_2^2}{Z_1 A_1^2} = \frac{4 Z_1 Z_2}{(Z_1 + Z_2)^2}$$

- Note : if $Z_1 = Z_2$ no energy is reflected and the impedances are said to be **matched**.

Example 3

A transverse harmonic force of peak value 0.3 N and frequency 5 Hz initiates waves of amplitude 0.1 m at one end of a very long string of linear density 0.01 kg/m. Determine the rate of energy transfer along the string and the wave velocity. (ans : $3\pi/20$ W and $30/\pi$ m/s)

solution

$$\text{rate of energy transfer } = \frac{1}{2}\rho c\omega^2 A^2$$

$$\because \text{ impedance Z} = \frac{F}{v} = \rho c$$

$$\text{Suppose } \ y = A\sin(\omega t - kx)$$

$$v = \dot{y} = \omega A\cos(\omega t - kx)$$

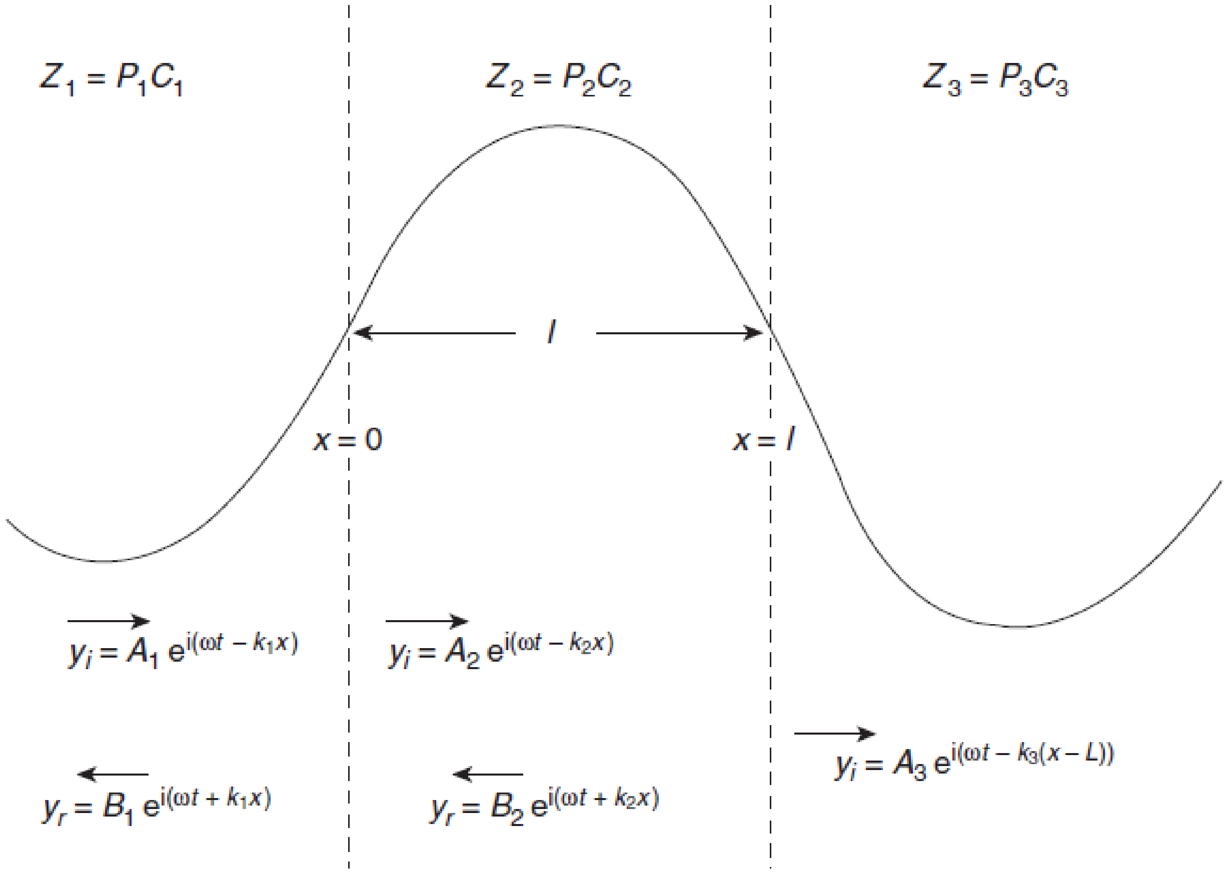
$$\therefore z = \rho c = \frac{F}{\omega A} = \frac{0.3}{(2\pi \times 5 \times 0.1)} = \frac{0.3}{\pi} \ \ \frac{\text{N}}{\text{m}} \cdot \text{s}$$

$$\therefore \text{ rate of energy transfer } = \frac{1}{2}\rho c\omega^2 A^2 = \frac{1}{2}\left(\frac{0.3}{\pi}\right)(2\pi \times 5)^2(0.1)^2$$

$$= \frac{3}{20}\pi \text{ W}$$

$$\text{and} \qquad c = \frac{Z}{\rho} = \left(\frac{0.3}{\pi}\right)\left(\frac{1}{0.01}\right) = \left(\frac{30}{\pi}\right) \ \ \frac{\text{m}}{\text{s}}$$

The matching of impedances



- Impedance matching represents a very important practical problem in the transfer of energy.
- The concept of impedance matching is to have the impedance change slowly.
- An appropriate length and impedance of another string between two mismatched strings will eliminate energy reflection and match the impedance.

The analysis of the matching impedances

- To comply with the condition of matching impedances, the following ratio has to be **unity**,

$$\frac{\text{Transmitted Energy}}{\text{Incident Energy}} = \frac{Z_3 A_3^2}{Z_1 A_1^2} = 1$$

- To derive the impedance and the length of the inserted piece of the string, the boundary conditions are that y and T(∂y/∂x) are continuous across the junctions x = 0 and x = *l*.

- This gives

$$\frac{\text{Transmitted Energy}}{\text{Incident Energy}} = \frac{Z_3 A_3^2}{Z_1 A_1^2} = \frac{1}{r_{13}} \frac{A_3^2}{A_1^2} = \frac{4 r_{13}}{(r_{13}+1)^2 \cos^2 k_2 l + (r_{12}+r_{23})^2 \sin^2 k_2 l}$$

- Where $r_{12} r_{23} = \frac{Z_1}{Z_2} \frac{Z_2}{Z_3} = \frac{Z_1}{Z_3} = r_{13}$

Conditions of impedance matching

- If the thickness of the coupling medium is chosen to be $\lambda_2/4$, the Z_2 can be written in terms of Z_1 and Z_3 as……………
- Recall the matching condition:

$$\frac{\text{Transmitted Energy}}{\text{Incident Energy}} = \frac{4r_{13}}{(r_{13}+1)^2\cos^2 k_2 l + (r_{12}+r_{23})^2 \sin^2 k_2 l} = 1$$

- By substituting $l = \lambda_2/4$ into the above equation and rearrange the equation by starting from

$$4r_{13} = (r_{12} + r_{23})^2 \quad or \quad 4\frac{Z_1}{Z_3} = \left(\frac{Z_1}{Z_2} + \frac{Z_2}{Z_3}\right)^2$$

- At the end after solving the quadratic equation of Z_2, we found that

$$Z_2 = \sqrt{Z_1 Z_3}$$

Standing waves on a string with fixed ends

- The displacement on the string at any point is given by $y = ae^{i(\omega t - kx)} + be^{i(\omega t + kx)}$
- With the boundary condition that y = 0 at x = 0 and x = *l* at all times, thus

$$y = (-2i)ae^{i(\omega t)}\sin kx$$

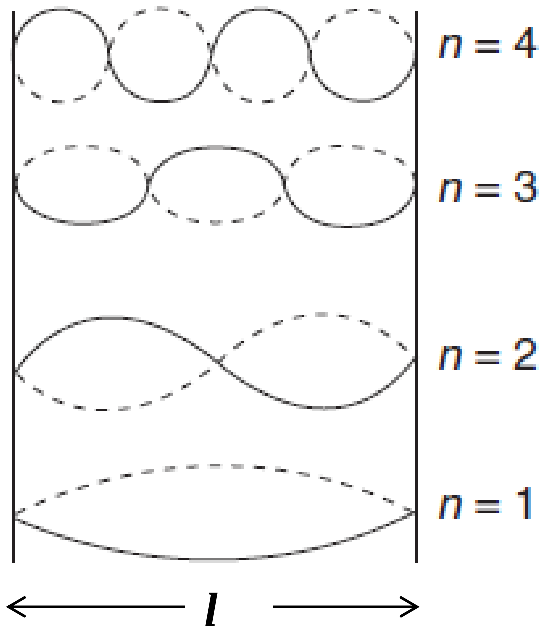
- The complete expression for the displacement of the ***nth* harmonic** is given by

$$y_n = 2a(-i)(\cos\omega_n t + i\sin\omega_n t)\sin\frac{\omega_n x}{c}$$

- This may be expressed as $y_n = (A_n\cos\omega_n t + B_n\sin\omega_n t)\sin\frac{\omega_n x}{c}$

Derivation of the nth normal mode standing wave function

- Recall the superposition of two travelling waves : $y = ae^{i(\omega t - kx)} + be^{i(\omega t + kx)}$
- By applying the 1st BC : @x = 0, y = 0; $0 = ae^{i\omega t} + be^{i\omega t}$; $\therefore\ b = -a$.
- This leads to $y = ae^{i(\omega t - kx)} - ae^{i(\omega t + kx)} = ae^{i\omega t}(-2i \sin kx)$
- By applying the 2nd BC : @ x = l, y = 0; 0 = $ae^{i\omega t}(-2i \sin kx)$.
- This leads to $kl = n\pi \to l = n\lambda/2$. This can be illustrated as



- Therefore, a proper form of the nth normal mode standing wave function

can be written as $y_n = 2a(-i)(\cos \omega_n t + i \sin \omega_n t) \sin \dfrac{\omega_n x}{c}$

and simplified to

$$y_n = \left(A_n \cos \omega_n t + B_n \sin \omega_n t\right) \sin \frac{\omega_n x}{c}$$

Example 4

- Show that the total energy of the *n*th normal mode of a standing wave with a length of *l* is given

$$E_n = \frac{1}{4} m\omega_n^2 (A_n^2 + B_n^2)$$

- Given that the wave function of *n*th normal is written as

$$y_n = (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{\omega_n x}{c}$$

Solution

- The energy in each harmonic is composed of kinetic and potential energy.

$$E_n(\text{kinetic} + \text{potential}) = E_n(\text{kinetic}) + E_n(\text{potential})$$

$$= \frac{1}{2}\int_0^l \rho \dot{y}_n^2 dx + \frac{1}{2} T \int_0^l \left(\frac{\partial y_n}{\partial x} \right)^2 dx$$

$$= \frac{1}{4} \rho l \omega_n^2 \left(A_n^2 + B_n^2 \right)$$

$$= \frac{1}{4} m \omega_n^2 \left(A_n^2 + B_n^2 \right)$$

- Where $y_n = \left(A_n \cos \omega_n t + B_n \sin \omega_n t \right) \sin \frac{\omega_n x}{c}$

Homework #5

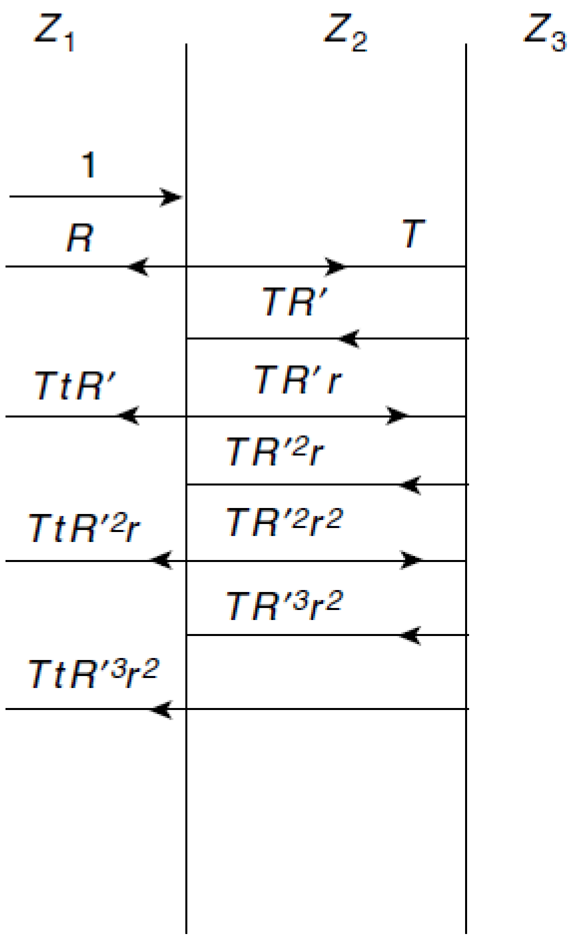
1. In the figure, media of impedances Z_1 and Z_3 are separated by a medium of intermediate impedance Z_2 and thickness $\lambda/4$ measured in this medium. A normally incident wave in the first medium has unit amplitude and the reflection and transmission coefficients for multiple reflections are shown. Show that the total reflected amplitude in medium 1 which is

$$R + tTR'(1 + rR' + r^2R'^2 \ldots)$$

is zero at $R = R'$ and show that this defines the condition

$$Z_2^2 = Z_1 Z_3$$

(Note that for zero total reflection in medium 1, the first reflection R is cancelled by the sum of all subsequent reflections.)



2. The relation between the impedance Z and the refractive index n of a dielectric is given by $Z = 1/n$. Light travelling in free space enters a glass lens which has a refractive index of 1.5 for a free space wavelength of 5.5×10^{-7} m. Show that reflections at this wavelength are avoided by a coating of refractive index 1.22 and thickness 1.12×10^{-7} m.